Modern approaches to quantum gravity

Solution 7

Fall 2025

1. Spinning fields

(a) Let's go through the Ward identities for translations, dilatations, rotations, and special conformal transformations.
Translational invariance requires

$$\langle j_\mu(x) j_\nu(y)\rangle = f_{\mu\nu}(x-y)$$

for some function $f_{\mu\nu}$ that depends only on $x-y$.
Rotational covariance requires $f_{\mu\nu}(x-y)$ to transform as a rank-2 tensor under rotations. The only independent symmetric rank-2 tensor structures available are $\delta_{\mu\nu}$ and $\frac{(x-y)_\mu (x-y)_\nu}{|x-y|^2}$. Thus, we can write the general form of $f_{\mu\nu}(x-y)$ as:

$$f_{\mu\nu}(x-y) = g(x-y)\left(\delta_{\mu\nu} + B\frac{(x-y)_\mu (x-y)_\nu}{|x-y|^2}\right)$$

where B is a constant to be determined, and g is an undetermined function. Dilatations with scaling dimension Δ require a power-law behaviour for g:

$$g(x-y) = \frac{1}{|x-y|^{2\Delta}} \tag{1}$$

Therefore

$$f_{\mu\nu}(x-y) = \frac{1}{|x-y|^{2\Delta}}\left(\delta_{\mu\nu} + B\frac{(x-y)_\mu (x-y)_\nu}{|x-y|^2}\right)$$

(b) We will derive the Ward identity for the special conformal transformation, as the other equations are very similar.
Since $\xi^\mu(x) \propto x$ and we set $y=0$, we only need to consider the transformation of $j_\mu(x)$. Using the following

$$\begin{aligned}\frac{\partial x'^\nu}{\partial x'^\mu} &= \delta^\nu_\mu + 2(b\cdot x)\delta^\nu_\mu + 2(b_\mu x^\nu - b^\nu x_\mu)\\ j_\mu(x') &= j_\mu(x) + (2(b\cdot x)x^\nu - x^2 b^\nu)\partial_\nu j_\mu(x)\end{aligned} \tag{2}$$

we can expand the tensor transformation formula to first order, and find $j'_\mu(x) = j_\mu(x) + \delta j_\mu(x)$, with

$$\delta j_\mu = (2\Delta(b\cdot x)\delta^\nu_\mu + 2(b_\mu x^\nu - b^\nu x_\mu))j_\nu + (2(b\cdot x)x^\nu - x^2 b^\nu)\partial_\nu j_\mu(x) \tag{3}$$

Plugging this into the Ward identity, we get the desired result

$$\begin{aligned}\langle \delta j_\mu(x) j_\nu(0)\rangle &= 0\\ \Big\{(2\Delta(b\cdot x) + 2(b\cdot x)x^\rho\partial_\rho + x^2 b^\rho\partial_\rho)\delta^\mu_\lambda + 2(b_\lambda x^\mu - b^\mu x_\lambda)\Big\}\langle j_\mu(x) j_\nu(0)\rangle &= 0\end{aligned} \tag{4}$$

(c) The first two properties from the first hint are immediate to check, so let us verify the third one.
From the first property we obtain that $\det\big(I^{\rho}_{\mu}\big) = \pm 1$, therefore

$$\left|\frac{\partial x'}{\partial x}\right| = \frac{|\det\big(I^{\rho}_{\mu}\big)|}{x^{2d}} = \frac{1}{x^{2d}} \tag{5}$$

Now, using the vector transformation under inversion we find

$$j'_{\mu}(x) = \left|\frac{\partial x'}{\partial x}\right|^{\frac{\Delta-1}{d}} \frac{I^{\nu}_{\mu}(x)}{x^2} j_{\nu}(x') = \frac{I^{\nu}_{\mu}(x)}{x^{2\Delta}} j_{\nu}(x') \tag{6}$$

Hence we need to check

$$\langle j'_{\mu}(x) j'_{\nu}(y)\rangle = \frac{I^{\rho}_{\mu}(x)}{x^{2\Delta}} \frac{I^{\sigma}_{\nu}(y)}{y^{2\Delta}} \langle j_{\rho}(x') j_{\sigma}(y')\rangle \doteq \langle j_{\mu}(x) j_{\nu}(y)\rangle \tag{7}$$

Equivalently,

$$\frac{I^{\rho}_{\mu}(x)}{x^{2\Delta}} \frac{I^{\sigma}_{\nu}(y)}{y^{2\Delta}} f_{\rho\sigma}(x'-y') \doteq f_{\mu\nu}(x-y) \tag{8}$$

Let us expand the LHS, using $y \to 0, y \to \infty$, and use the fact that $\frac{1}{|x'-y'|^{2\Delta}} = \frac{x^{2\Delta} y^{2\Delta}}{|x-y|^{2\Delta}}$

$$\begin{aligned}
\frac{I^{\rho}_{\mu}(x)}{x^{2\Delta}} \frac{I^{\sigma}_{\nu}(y)}{y^{2\Delta}} f_{\rho\sigma}(x'-y') &= \frac{I^{\rho}_{\mu}(x)}{x^{2\Delta}} \frac{I^{\sigma}_{\nu}(y)}{y^{2\Delta}} \frac{I_{\rho\sigma}(x'-y')}{|x'-y'|^{2\Delta}} \\
&= I^{\rho}_{\mu}(x) I^{\sigma}_{\nu}(y) \frac{I_{\rho\sigma}(y')}{|x-y|^{2\Delta}} \Big(1 + \mathcal{O}(x'/y')\Big) \\
&= I^{\rho}_{\mu}(x) I^{\sigma}_{\nu}(y) \frac{I_{\rho\sigma}(y)}{|x-y|^{2\Delta}} \Big(1 + \mathcal{O}(x'/y')\Big) \\
&= \frac{I_{\mu\nu}(x)}{|x-y|^{2\Delta}} \Big(1 + \mathcal{O}(x'/y')\Big) \\
&= \frac{I_{\mu\nu}(x-y)}{|x-y|^{2\Delta}} \Big(1 + \mathcal{O}(x'/y')\Big) \\
&\lim_{y\to 0} f_{\mu\nu}(x-y)
\end{aligned} \tag{9}$$

Note that this derivation would have proceeded similarly even if you tried to used a general value of y.

(d) Consider $y = 0$. Then

$$\partial_{\mu}\langle j^{\mu}(x) j^{\nu}(0)\rangle \propto -2\Delta x_{\mu} I^{\mu\nu}(x) + x^2 \partial_{\mu} I^{\mu\nu} \propto (\Delta - d + 1) = 0 \tag{10}$$

Hence $\Delta = d - 1$. This ensures that the conserved charge $Q \equiv \int d^{d-1}x \; n^{\mu} j_{\mu}$ is scale-invariant.

(e) The proof is similar to point 1 and 2, but more tedious. The efficient way to do this is via the embedding space formalism.
Analogously to point 3, we require

$$\partial_{\mu}\langle T^{\mu\nu}(x) T^{\rho\sigma}(0)\rangle \propto (\Delta - d) = 0 \tag{11}$$

Hence $\Delta = d - 1$. This ensures that the charges generating conformal transformations are scale-invariant.

2. Introduction to the embedding space formalism

(a) Using that

$$dP_0^2 = dP_{d+1}^2 = x_\mu x_\nu dx^\mu dx^\nu \quad dP^\mu = dx^\mu \tag{12}$$

it is straightforward to see that

$$ds^2 = -dP_0^2 + dP_{d+1}^2 + (dP^\mu)^2 = \delta_{\mu\nu} dx^\mu dx^\nu \tag{13}$$

(b) Let us take

$$P^0 = \Omega(x)\frac{1+x^2}{2} \quad P^{d+1} = \Omega(x)\frac{1-x^2}{2} \quad P^\mu = \Omega(x)x^\mu \tag{14}$$

Then,

$$dP^0 = (\partial_\mu \Omega)\frac{1+x^2}{2}dx^\mu + \Omega x_\mu dx^\mu \tag{15}$$

$$dP^{d+1} = (\partial_\mu \Omega)\frac{1-x^2}{2}dx^\mu - \Omega x_\mu dx^\mu \tag{16}$$

$$dP^\mu = (\partial_\nu \Omega)x^\mu dx^\nu + \Omega dx^\mu \tag{17}$$

Then by expanding explicitly all terms, we find that

$$ds^2 = -dP_0^2 + dP_{d+1}^2 + (dP^\mu)^2 = \Omega^2 \delta_{\mu\nu} dx^\mu dx^\nu \tag{18}$$

(c) Let us start by the two-point function. By Lorentz invariance, $\langle \mathcal{O}(P_1)\mathcal{O}(P_2)\rangle$ can only be a function of scalar invariants built from P_1 and P_2, i.e. P_1^2, P_2^2 and $P_1 \cdot P_2$. Since we are on the light-cone, $P_1^2 = P_2^2 = 0$, thus

$$\langle \mathcal{O}(P_1)\mathcal{O}(P_2)\rangle = f(P_1 \cdot P_2) \tag{19}$$

Since this needs to satisfy the homogeneity property, it can only be a power-law,

$$f(P_1 \cdot P_2) = \text{const}|P_1 \cdot P_2|^\alpha \tag{20}$$

where α is determined by the homogeneity property

$$f(\lambda P_1 \cdot P_2) = \lambda^{-2\Delta} f(P_1 \cdot P_2) \implies \alpha = -\Delta \tag{21}$$

Thus

$$\langle \mathcal{O}(P_1)\mathcal{O}(P_2)\rangle = \frac{\text{const}}{|P_1 \cdot P_2|^\Delta} \tag{22}$$

Note that on the physical section,

$$P_1(x_1)\cdot P_2(x_2) = -\left(\frac{1+x_1^2}{2}\frac{1+x_2^2}{2}\right) + \left(\frac{1-x_1^2}{2}\frac{1-x_2^2}{2}\right) + x_1 \cdot x_2 = -\frac{1}{2}(x_1 - x_2)^2 \tag{23}$$

Thus,

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = \frac{\text{const}'}{|x_1 - x_2|^{2\Delta}} \tag{24}$$

For the 3-point function, we have

$$\langle \mathcal{O}(P_1)\mathcal{O}(P_2)\mathcal{O}(P_3)\rangle = f(P_1 \cdot P_2, P_1 \cdot P_3, P_2 \cdot P_3) \tag{25}$$

Note that the correlator is invariant under permutations of $(1, 2, 3)$, thus f should be invariant under permutations of its 3 arguments. Also, the homogeneity property implies again a power-law. Thus,

$$\langle \mathcal{O}(P_1)\mathcal{O}(P_2)\mathcal{O}(P_3)\rangle = (|P_1 \cdot P_2||P_1 \cdot P_3||P_2 \cdot P_3|)^\alpha \tag{26}$$

Finally, α is again determined from scaling which implies

$$\lambda^{6\alpha} = \lambda^{-3\Delta} \implies \alpha = -\frac{\Delta}{2} \tag{27}$$

Thus,

$$\langle \mathcal{O}(P_1)\mathcal{O}(P_2)\mathcal{O}(P_3)\rangle = \frac{\text{const}}{|P_1 \cdot P_2|^{\Delta/2}|P_1 \cdot P_3|^{\Delta/2}|P_2 \cdot P_3|^{\Delta/2}} \tag{28}$$

On the physical section, using $P_1(x_1) \cdot P_2(x_2) \propto (x_1 - x_2)^2$ as previously, we obtain

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\rangle = \frac{\text{const}'}{|x_{12}|^\Delta |x_{13}|^\Delta |x_{23}|^\Delta} \tag{29}$$

(d) For the two-point function of vector operators $\langle \mathcal{O}^A(P_1)\mathcal{O}^B(P_2)\rangle$, we impose Lorentz covariance by noting that the result must be built out of $P_1^A\ P_2^B$ or η^{AB}, with Lorentz contractions that can only be made of $P_1 \cdot P_2$, namely

$$\langle \mathcal{O}^A(P_1)\mathcal{O}^B(P_2)\rangle = a_1\eta^{AB} f_1(P_1 \cdot P_2) + a_2 P_1^B P_2^A f_2(P_1 \cdot P_2) + a_3 P_1^A P_2^B f_3(P_1 \cdot P_2) \tag{30}$$

where f_1, f_2 and f_3 are power-laws because of homogeneity. Thus,

$$\langle \mathcal{O}^A(P_1)\mathcal{O}^B(P_2)\rangle = a_1\eta^{AB}|P_1 \cdot P_2|^{\alpha_1} + a_2 P_1^B P_2^A |P_1 \cdot P_2|^{\alpha_2} + a_3 P_1^A P_2^B |P_1 \cdot P_2|^{\alpha_3} \tag{31}$$

Homogeneity implies

$$\alpha_1 = -\Delta \quad \alpha_2 = \alpha_3 = -(\Delta + 1) \tag{32}$$

Thus,

$$\langle \mathcal{O}^A(P_1)\mathcal{O}^B(P_2)\rangle = \frac{a_1'(P_1 \cdot P_2)\eta^{AB} + a_2' P_2^A P_1^B + a_3' P_1^A P_2^B}{(-2P_1 \cdot P_2)^{\Delta+1}} \tag{33}$$

We also impose that

$$(P_1)_A\langle \mathcal{O}^A(P_1)\mathcal{O}^B(P_2)\rangle = (P_2)_B\langle \mathcal{O}^A(P_1)\mathcal{O}^B(P_2)\rangle = 0 \tag{34}$$

which implies $a_2' = -a_1'$. Thus,

$$\langle \mathcal{O}^A(P_1)\mathcal{O}^B(P_2)\rangle = \text{const.}\frac{(P_1 \cdot P_2)\eta^{AB} - P_2^A P_1^B}{(-2P_1 \cdot P_2)^{\Delta+1}} + a_3' \frac{P_1^A P_2^B}{(-2P_1 \cdot P_2)^{\Delta+1}} \tag{35}$$

The last term is irrelevant since vector operators are identified by the redundancy $\mathcal{O}^A(P) \sim \mathcal{O}^A(P) + P^A\lambda(P)$.

(e) To go to physical space,

$$\langle \mathcal{O}_\mu(x_1)\mathcal{O}_\nu(x_2)\rangle = \frac{\partial P_1^A(x_1)}{\partial x_1^\mu}\frac{\partial P_2^B(x_2)}{\partial x_1^\nu}\langle \mathcal{O}^A(P_1)\mathcal{O}^B(P_2)\rangle \tag{36}$$

Noting that

$$\frac{\partial P_1^A(x_1)}{\partial x_1^\mu}\frac{\partial P_2^A(x_2)}{\partial x_2^\nu}\eta_{AB} = \eta_{\mu\nu} \tag{37}$$

$$\frac{\partial P_1^A(x_1)}{\partial x_1^\mu}P_{2,A}(x_2) = -(x_{1,\mu} - x_{2,\mu}) \tag{38}$$

$$P_1 \cdot P_2 = -\frac{1}{2}|x_1 - x_2|^2 \tag{39}$$

This gives

$$\frac{\partial P_1^A(x_1)}{\partial x_1^\mu}\frac{\partial P_2^B(x_2)}{\partial x_1^\nu}\left(\eta_{AB} - \frac{(P_2)_A(P_1)_B}{(P_1 \cdot P_2)}\right) = I_{\mu\nu}(x_1 - x_2) \tag{40}$$

Thus,

$$\langle \mathcal{O}_\mu(x_1)\mathcal{O}_\nu(x_2)\rangle = \text{const}'\frac{I_{\mu\nu}(x_1 - x_2)}{|x - y|^{2\Delta}} \tag{41}$$

as derived in exercise 1.